UNIT - V

ASSIGNMENT PROBLEM.

Introduction

Definition of A.P [The assignment problem is a particular case of the transportation problem in which the objective is to assign a number of tasks (Jobs or origin or sources) to an equal number of facilities (machines or persons or destinations) at a minimum cost [or maximum profit]].

2m [The assignment problem can be stated in the form of $m \times n$ matrix (c_{ij}) called a cost matrix (or) Effectiveness matrix where c_{ij} is the cost of assigning i^{th} machine to the j^{th} job.

$$
\begin{array}{cc}
 & \text{Jobs} \\
 & \begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \end{array} \\
\text{Machines} \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ \vdots \\ m \end{array} &
\begin{array}{ccccc}
c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\
c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\
c_{31} & c_{32} & c_{33} & \cdots & c_{3n} \\
\vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & \vdots & & \vdots \\
c_{m1} & c_{m2} & c_{m3} & \cdots & c_{mn}
\end{array}
\end{array}
$$

] 2m

Mathematical formulation of an assignment problem

2m [Consider an assignment problem of assigning n jobs to n machines (one job to one machine). Let c_{ij} be the unit cost of assigning i^{th} machine to the j^{th} job and let $x_{ij} = \begin{cases} 1 & \text{if } j^{th} \text{ job is assigned to } i^{th} \text{ mach} \\ 0 & \text{if } j^{th} \text{ job is not assigned to } i^{th} \text{ machine.} \end{cases}$

2m [The assignment model is then given by the following L.P.P.

Minimize $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1, 2 \ldots n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1, 2 \ldots n$$

and $x_{ij} = 0$ (or) 1.]

2m/5m Difference between the transportation problem and the assignment problem:

	T.P.	A.P.
(a)	supply at any source may be any positive quantity a_i	supply at any source (machine) will be 1 i.e., $a_i = 1$.
(b)	Demand at any destination may be any positive quantity b_j	Demand at any destination (job) will be 1 i.e., $b_j = 1$.
(c)	One or more source to any number of destinations	One source (machine) to only one destination (job).

10M Assignment Algorithm (or) Hungarian Method.

First check whether the number of rows is equal to the number of columns, If it is so, the assignment problem is said to be balanced. Then proceed to step 1. If it is not balanced, then it should be balanced before applying the algorithm. The method of Balancing is checked in ~~Sec 5.6~~

Step : 1

Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

step : 2

Subtract the smallest cost element of each column from all the elements in the column of the resulting cost matrix obtained by step : 1

step : 3 (Assigning the zeros)

a) Examine the rows successively until a row with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it, cross all other zeroes in the column of this encircled zero, as these will not be considered for any future assignment. Continue in this way until all the rows have been examined.

b) Examine the columns successively until a column with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it and cross any other zero in its row. Continue until all the columns have been examined.

step : 4 [Apply Optimal Test]

a) If each row and each column contain exactly one encircled zero, then the current assignment is optimal.

b) If atleast one row/column is without an assignment [ie., if there is atleast one row/column is without one encircled zero) then the current assignment is not optimal. Go to step 5.

Step 5:

Cover all the zeros by drawing a minimum number of straight line as follow

a) Mark (✓) the rows that do not have an assignment.

c) ~~b)~~ Mark (✓) the rows (not already marked) that have an assignment in marked columns.

b) Mark (✓) the columns (not already marked) that have zeros in marked rows

d) Repeat (b) & (c) until no more marking is required.

e) Draw lines through all unmarked rows and marked columns. If the number of these lines is equal to the order of the matrix then it is an optimal solution otherwise not.

Step 6:

Determine the smallest cost element not covered by the straight lines, Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

Step 7:

Repeat steps (1) to (6), until an optimum assignment is attained.

<u>Note</u> (1)

In case some rows or columns contain more than one zero, encircle any unmarked zero arbitrarily and cross all other zeros in its column or row. Proceed in this way until no zero is left unmarked or encircled.

② The above assignment algorithm is only for minimization problems.

③ If the given assignment problem is of maximization type, convert it to a minimization assignment problem by max z = − min (−z) and multiply all the given cost elements by −1 in the cost matrix and then solve by assignment algorithm.

④ Some times, a final cost matrix contains more than required number of zeros at independent positions. This implies that there is more than one optimal solution (multiple optimal solutions) with the same optimum assignment cost.

2m Unbalanced Assignment Models:

[If the number of rows is not equal to the number of columns in the cost matrix of the given assignment problem, then the given assignment problems is said to be unbalanced].

First convert the unbalanced assignment problem into a balanced one by adding dummy rows or dummy columns with zero cost elements in the cost matrix depending upon whether $m < n$ or $m > n$ and then solve by the usual method.

Maximization case in Assignment Problems.

In an assignment problem, we may have to deal with maximization of an objective function. For example, we have to assign persons to jobs in such a way that the total profit is maximized. The maximization problem has to be converted into an equivalent minimization problem and then solved by the usual Hungarian method.

The conversion of the maximization problem into an equivalent minimization problem can be done by any one of the following methods:

i) Since max $z = -\min(-z)$, multiply all the costs elements c_{ij}, of the cost matrix by -1.

ii) Subtract all the cost elements c_{ij} of the cost matrix from the highest cost element in that cost matrix.

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ASSIGNMENT PROBLEM

In an assignment problem, the number of jobs available equals the number of machines. Each job is assigned to only one machine and each machine is assigned only one job such that the total cost is a minimum. That is, the assignment is on a one-to-one basis.

This kind of problem is faced in assigning planes / crew to commercial flights, buses, conductors and drivers to various routes, managers to different branches, etc.

When there are more jobs than machines, excess jobs are not done. When the number of machines is more than the number of jobs, excess machines remain idle.

For example, consider the four jobs A, B, C and D and the four machines I, II, III and IV. Let the unit costs be as follows :

Jobs	Machines			
	I	II	III	IV
A	30	25	26	28
B	26	32	24	20
C	20	22	18	27
D	23	20	21	19

That is, let the cost of assigning job A to machine I be 30, the cost of assigning job A to machine II be 25 and so on. The objective is to find the choice of assigning the four jobs to the four machines such that each machine gets only one job and the total cost is the minimum.

An assignment problem is a special case of transportation problems in which

number of origins (jobs) n = number of destinations (machines), n

number of units of i^{th} source allocated to j^{th} destination,

x_{ij} = 1 if the j^{th} machine is assigned the i^{th} job
= 0 if the j^{th} machine is not assinged the i^{th} job

(x_{ij} is a dichotomy which takes a value 1 or 0)

$$a_i = \sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1, 2, \ldots, n$$

$$b_j = \sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1, 2, \ldots, n$$

The following is a model assignment table :

Jobs	Machines					
	M_1	M_2	...	M_j	...	M_n
J_1	c_{11}	c_{12}	...	c_{1j}	...	c_{1n}
J_2	c_{21}	c_{22}	...	c_{2j}	...	c_{2n}
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	...	$\vdots$
J_i	c_{i1}	c_{i2}	...	c_{ij}	...	c_{in}
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	...	$\vdots$
J_n	c_{n1}	c_{n2}	...	c_{nj}	...	c_{nn}

c_{ij} is the cost when j^{th} machine is assigned i^{th} job.

Mathematical Model

An assignment problem is a special case of LPP. Its mathematical model :

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1, 2, \ldots, n$$

and (non-negativity restriction) x_{ij} = 0 or 1

Hungarian Method

Hungarian Method is also called **Flood's Method** or **Reduced Matrix Method.** It saves time substantially over the other methods. It reduces the cost matrix rapidly. A set of n zeros, one in each row and each column, is obtained. It is the optimal solution.

The following are the steps:

Step 1 : (A necessary and sufficient condition)

Check whether it is a balanced problem. That is, find out whether the number of rows and the number of columns are equal in the cost matirx.

If not, add necessary number of rows (called, dummy rows) or necessary number of columns (called, dummy columns) as required to make them equal in number. Each (cost) element of dummy rows and dummy columns is zero.

Step 2 : (Reduce the elements of each row)

Identify the smallest cost element in each row and subtract it from all the elements of the row and get the reduced matrix.

Step 3 : (Reduce the elements of each column)

Identify the smallest cost element in each column of the reduced matrix obtained in step 2 and subtract it from all the elements of the column and get the reduced matrix.

Thus, each row as well as each column has at least one zero.

Step 4 : (Make the assignments)

(a) Examine the rows one by one. If there is only one zero in a row, make the assignment by drawing a square around the zero $\boxed{0}$ and cross out all other zeros ($\cancel{0}$) in its column.

(b) Examine the columns one by one. If there is only one zero in a column, make the assignment by drawing a square around the zero $\boxed{0}$ and cross out all other zeros ($\cancel{0}$) in its row.

(c) If any row and / or any column has two or more zeros, assign arbitrarily any one of those zeros and cross out all other zeros in that row / column.

(d) Repeat (a) through (c) above successively until all possible assignments '$\boxed{0}$' and cross outs '(~~0~~)' are over.

Step 5 : (Test for optimality)

If each row as well as each column has exactly one assignment $\boxed{0}$, the current assignment is optimal.

If there is at least one row or column without an assignment, go to the next step.

Step 6 : Draw the minimum number of horizontal and / or vertical lines and cover all the zeros.

If necessary, those rows and columns may be identified as follows :

(a) Tick (✓) the rows which do not have any assignment.

(b) Tick (✓) the columns which have crossed zeros in the ticked rows.

(c) Tick (✓) the rows which have assigned zeros in the ticked columns.

(d) Repeat (b) and (c) above until no more ticking is possible.

(e) Draw lines through *all unticked rows and ticked columns.*

Step 7 : Identify the smallest element of the reduced matrix from the elements which are not covered by the lines. Subtract that number from all the uncovered elements and add it to all the elements at the intersections of the lines. Don't change the other elements which lie along the lines.

Step 8 : Go to step 4 and repeat the steps until an optimal solution is obtained.

Note : 1. Alternative optimal solutions exist when the final assignment matrix (step 4(c)) contains more than the required number of zero elements.

2. A maximization problem can be solved as above after changing the given profit matrix as follows :

From the largest element of the profit matrix, every element is subtracted to get the opportunity loss matrix.

(or)

All the elements of the profit matrix is multiplied by –1.

Example 1 : Solve the following minimal assignment problem by Hungarian method.

Machines

Jobs		1	2	3	4
	A	9	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

(B.Sc. Bharathidasan, A 03)

Solution :

Step 1 : The number of rows = The number of columns. It is a balanced problem. It is solved as follows :

Setp 2 : Reduce the elements of each row :

9, 4, 15 and 10 are the smallest elements in the first, second, third and fourth rows respectively. 9 is subtracted from the elements of the first row, 4, from those of the second row, ... The reduced matrix (a) is given below.

Machines

Jobs		1	2	3	4
	A	0	17	8	2
	B	9	24	0	22
	C	23	4	3	0
	D	9	16	14	0

(a)

Machines

Jobs		1	2	3	4
	A	0	13	8	2
	B	9	20	0	22
	C	23	0	3	0
	D	9	12	14	0

(b)

Step 3 : Reduce the elements of each column :

0, 4, 0 and 0 are the smallest elements in the first, second, third and fourth columns respectively. 0 is substracted from the elements of the first column, 4, from those of the second column, . . . The reduced matrix (b) is given in the previous page.

Step 4 : Make the assignments :

In the first, second and fourth rows, there is only one zero in each row. Squares are drawn around them. The other zero in the fourth column is crossed. While considering columnwise there is only one zero in the second column. A square is drawn around it.

Machines

Jobs		1	2	3	4
	A	$\boxed{0}$	13	8	2
	B	9	20	$\boxed{0}$	22
	C	23	$\boxed{0}$	3	$\cancel{0}$
	D	9	12	14	$\boxed{0}$

Step 5 : Test the optimality :

Each row as well as each column has exactly one assignment $\boxed{0}$. The optimal assignment has been found.

The assignment schedule :

A → 1 ; B → 3; C → 2 and D → 4.

The minimum assignment cost :

Min. Z = 9 + 4 + 19 + 10 = 42

Example 2 : Solve the following minimal assignment problem :

Worker

Job		A	B	C	D
	1	41	72	39	52
	2	22	29	49	65
	3	27	39	60	51
	4	45	50	48	52

(B.Com. Madras, O 02)

Solution :

Step 1 : The number of rows = The number of columns. It is a balanced problem. It is solved as follows :

Step 2 : Reduce the elements of each row :

39, 22, 27 and 45 are the least elements in the first, second, third and fourth rows respectively. 39 is subtracted from the elements of the first row, 22, from those of the second row, . . . The reduced matrix (a) is given below.

Worker

Job	A	B	C	D
1	2	33	0	13
2	0	7	27	43
3	0	12	33	24
4	0	5	3	7

(a)

Worker

Job	A	B	C	D
1	2	28	0	6
2	0	2	27	36
3	0	7	33	17
4	0	0	3	0

(b)

Step 3 : Reduce the elements of each column :

0, 5, 0 and 7 are the least elements in the first, second, third and fourth columns respectively. 5 is subtracted from the elements of the second column and 7, from those of the fourth column. The reduced matrix (b) is given above.

Step 4 : Make the assignments :

There is only one zero in the first row as well as the second. Squares are put around them. The other zeros, in the first column are crossed. There is no zero in the third row and there are two zeros in the fourth row.

There is only one zero in the second column. A square is put around it. The other zero in the fourth row is crossed. They are shown in the matrix (c) given below.

Worker

Job	A	B	C	D
1	2	28	$\boxed{0}$	6
2	$\boxed{0}$	2	27	36
3	$\cancel{0}$	7	33	17
4	$\cancel{0}$	$\boxed{0}$	3	$\cancel{0}$

(c)

Worker

Job	A ✓	B	C	D	
1	2	28	$\boxed{0}$	6	
2	$\boxed{0}$	2	27	36	✓
3	$\cancel{0}$	7	33	17	✓
4	$\cancel{0}$	$\boxed{0}$	3	$\cancel{0}$	

(d)

Step 5 : Test for optimality :

The third row doesn't have an assignment; so does the fourth column.

Step 6 : By drawing lines along the first and the fourth rows and the first column, all the zeros are covered.

If there is any difficulty in determining as above to draw the minimum number of lines for covering the zeros, the following procedure is followed and the resultant matrix (d) is given above.

(a) Third row has no assignment. It is ticked.

(b) In the third row, crossed zero is in the first column. The first column is, hence, ticked.

(c) In the first column, assigned zero is in the second row. The second row is, hence, ticked.

(d) There is no crossed zero in the second row. Hence, (b) and (c) could not be repeated.

(e) The first and the fourth rows are unticked. The first column is ticked. The lines are drawn along them.

Step 7 : The elements which are not covered by the lines are 2, 27, 36, 7,33 and 17. 2 is the smallest among them. So, 2 is subtracted from them and 2 is added to the elements 2 and 0 which are at the intersections of the lines. The other elements 28, 0, 6, 0, 0, 0, 3 and 0 are left unchanged. The revised matrix (e) is given below.

Worker

Job	A	B	C	D
1	4	28	0	6
2	0	0	25	34
3	0	5	31	15
4	2	0	3	0

(e)

Worker

Job	A	B	C	D
1	4	28	$\boxed{0}$	6
2	$\cancel{0}$	$\boxed{0}$	25	34
3	$\boxed{0}$	5	31	15
4	2	$\cancel{0}$	3	$\boxed{0}$

(f)

Step 8 : The assignments are made as in step 4 and shown in (f) in the previous page.

In the first and the third rows, there is only one zero each. Squares are put around them. The other zero in the first column is crossed. While considering columnwise, there is only one zero in the fourth column. A square is put around it. The other zero in the fourth row is scored out. There is only one zero in the second column at this stage. A square is put around it.

Therefore, the assignment schedule is

$1 \rightarrow C; 2 \rightarrow B; 3 \rightarrow A$ and $4 \rightarrow D$

The assignment cost is

min. $Z = 39 + 29 + 27 + 52 = 147$

Example 3 : (Alternative optimum solutions)

Solve the minimal assignment problem :

	I	II	III	IV
A	30	25	26	28
B	26	32	24	20
C	20	22	18	27
D	23	20	21	19

(B.Com. Madras, A 03)

Solution :

Step 1 : The number of rows = The number of columns. It is a balanced problem. It is solved as follows :

Step 2 : Reduce the elements of each row :

25, 20, 18 and 19 are the least elements in the first, second, third and fourth rows respectively. 25 is subtracted from the elements of the first row, 20, from those of the second row . . . and the reduced matrix (a) is obtained.

	I	II	III	IV
A	5	0	1	3
B	6	12	4	0
C	2	4	0	9
D	4	1	2	0

(a)

	I	II	III	IV
A	3	0	1	3
B	4	12	4	0
C	0	4	0	9
D	2	1	2	0

(b)

Step 3 : Reduce the elements of each column :

2, 0, 0 and 0 are the least elements in the first, second, third and fourth columns respectively. 2 is subtracted from the elements of the first column. The reduced matrix (b) is given above.

Setp 4 : Make the assignments :

There is only one zero each in the first two rows. Squares are put around them. The other zero in the fourth column is crossed. While considering columnwise, there is only one zero in the first column. A square is put around it. The other zero in the third row is crossed. They are shown in the matrix (c) given below.

	I	II	III	IV
A	3	$\boxed{0}$	1	3
B	4	12	4	$\boxed{0}$
C	$\boxed{0}$	4	$\cancel{0}$	9
D	2	1	2	$\cancel{0}$

(c)

	I	II	III	IV	
				✓	
A	3	$\boxed{0}$	1	3	
B	4	12	4	$\boxed{0}$	✓
C	$\boxed{0}$	4	$\cancel{0}$	9	
D	2	1	2	$\cancel{0}$	✓

(d)

Step 5 : Test for optimality :

The fourth row as well as the third column has no assignment.

Step 6 : The minimum number of lines to be drawn to cover the zeros is three (The first row, the third row and the fourth column). Refer to the matrix (d) above.

It may be done by the following method also.

(a) The fourth row has no assignment. It is ticked.

(b) It has the crossed zero in the fourth column. The
ırth column in ticked.

(c) The fourth column has assigned zero in the second
w. The second row is ticked.

(d) No repetition of (b) and (c) is possible as there is no
ossed zero in the second row.

(e) The first and the third rows are unticked. The fourth
lumn is ticked. Hence, lines are drawn along them.

Step 7 : In the matrix (d), the uncovered elements are 12, 4, 2, 1 and 2. The least among them is 1. Hence, 1 is ıbtracted from them and is added to 3 and 9 which are at ıe intersections of the lines. The other elements along the nes 3, 0, 1, 0, 0, 4, 0 and 0 remain unaltered. They are iven in the revised matrix (e) below.

$$\begin{array}{ccccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \text{A} & 3 & 0 & 1 & 4 \\ \text{B} & 3 & 11 & 3 & 0 \\ \text{C} & 0 & 4 & 0 & 10 \\ \text{D} & 1 & 0 & 1 & 0 \end{array} \qquad \begin{array}{ccccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \text{A} & 3 & \boxed{0} & 1 & 4 \\ \text{B} & 3 & 11 & 3 & \boxed{0} \\ \text{C} & \boxed{0} & 4 & \cancel{0} & 10 \\ \text{D} & 1 & \cancel{0} & 1 & \cancel{0} \end{array}$$

(e) (f)

Setp 8 : The assignments are made as in step 4 and ;hown in the matrix (f) above.

There is only one zero each in the first and the second rows. Squares are put around those zeros. The other zeros in the second and fourth columns are crossed. There are two zeros in the third row. No zero is available in the fourth row. In the first column, there is one zero. A square is put around it. The other zero in the third row is crossed.

Neither the fourth row nor the third column has any assignment.

As in step 6, by drawing lines across the third row, the second column and the fourth column, all the zeros are covered. Refer to the matrix (g) overleaf.

$$
\begin{array}{c|cccc}
 & \text{I} & \text{II} & \text{III} & \text{IV} \\
\hline
\text{A} & 3 & \boxed{0} & 1 & 4 \\
\text{B} & 3 & 11 & 3 & \boxed{0} \\
\text{C} & \boxed{0} & 4 & \cancel{0} & 10 \\
\text{D} & 1 & \cancel{0} & 1 & \cancel{0}
\end{array}
\qquad
\begin{array}{c|cccc}
 & \text{I} & \text{II} & \text{III} & \text{IV} \\
\hline
\text{A} & 2 & 0 & 0 & 4 \\
\text{B} & 2 & 11 & 2 & 0 \\
\text{C} & 0 & 5 & 0 & 11 \\
\text{D} & 0 & 0 & 0 & 0
\end{array}
$$

(g) (h)

The following is similar to step 7. 3, 1, 3, 3, 1 and 1 are the uncovered elements. The smallest among them is 1. Therefore, 1 is subtracted from all of them and is added to 4 and 10 which are at the intersections of the lines. The other elements 0, 4, 11, 0, 0, 0, 0 and 0 are left unchanged. The resultant matrix is (h) above.

In the second row, there is only one zero. A square is put around it. The other zero in the fourth column is crossed. The other rows and the other columns have more than one zero. Hence, the assignments are made arbitrarily as follows:

1.

$$
\begin{array}{c|cccc}
 & \text{I} & \text{II} & \text{III} & \text{IV} \\
\hline
\text{A} & 2 & \boxed{0} & \cancel{0} & 4 \\
\text{B} & 2 & 11 & 2 & \boxed{0} \\
\text{C} & \boxed{0} & 5 & \cancel{0} & 11 \\
\text{D} & \cancel{0} & \cancel{0} & \boxed{0} & \cancel{0}
\end{array}
$$

A → II; C → I; D → III. B → IV.

The assignment cost : 25 + 20 + 21 + 20 = 86

(or)

2.

$$
\begin{array}{c|cccc}
 & \text{I} & \text{II} & \text{III} & \text{IV} \\
\hline
\text{A} & 2 & \boxed{0} & \cancel{0} & 4 \\
\text{B} & 2 & 11 & 2 & \boxed{0} \\
\text{C} & \cancel{0} & 5 & \boxed{0} & 11 \\
\text{D} & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0}
\end{array}
$$

A → II; C → III; D → I. B → IV.

The assignment cost : 25 + 18 + 23 + 20 = 86

(or)

3.

	I	II	III	IV
A	2	$\cancel{0}$	$\boxed{0}$	4
B	2	11	2	$\boxed{0}$
C	$\boxed{0}$	5	$\cancel{0}$	11
D	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

A → III; C → I; D → II. B → IV.

The assignment cost : 26 + 20 + 20 + 20 = 86

The assignment costs are the least and equal for the above three alternative optimum solutions.

Example 4 : (Unbalanced Assignment Problem)

There are 4 jobs and 5 machines. The time (in hours) for each job on every machine is given in the following table. Find the optimal assignment schedule.

	Machine				
Job	I	II	III	IV	V
A	20	21	14	12	18
B	17	21	20	24	22
C	15	16	19	22	24
D	23	25	21	20	17

Solution :

Step 1 : The number of rows = 4; the number of columns = 5. It is an unbalanced assignment problem.

Hence, one dummy row with elements, zeros, is added and shown in (a) below

	Machine				
Job	I	II	III	IV	V
A	20	21	14	12	18
B	17	21	20	24	22
C	15	16	19	22	24
D	23	25	21	20	17
Dummy	0	0	0	0	0

(a)

	Machine				
Job	I	II	III	IV	V
A	8	9	2	0	6
B	0	4	3	7	5
C	0	1	4	7	9
D	6	8	4	3	0
Dummy	0	0	0	0	0

(b)

Step 2 : Reduce the elements of each row :

12, 17, 15, 17 and 0 are the least elements in the first, second, third, fourth and fifth rows respectively. 12 is subtracted from the elements of the first row, 17, from those of the second row, . . . The resulting matrix (b) is given overleaf.

Step 3 : Reduce the elements of each column :

0 is the least element in every column. When 0 is subtracted from the elements, there is no change.

Step 4 : Make the assignments :

There is only one zero each in the first, second and fourth rows. Squares are put around them. The other zeros in the fourth, first and fifth columns are crossed. There is only one zero in the second column. A square is put around it. The other zero in the fifth row is crossed. The resultant matrix (c) is given below.

Machine

Job	I	II	III	IV	V
A	8	9	2	$\boxed{0}$	6
B	$\boxed{0}$	4	3	7	5
C	$\cancel{0}$	1	4	7	9
D	6	8	4	3	$\boxed{0}$
Dummy	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

(c)

Machine

Job	I ✓	II	III	IV	V	
A	8	9	2	$\boxed{0}$	6	
B	$\boxed{0}$	4	3	7	5	✓
C	$\cancel{0}$	1	4	7	9	✓
D	6	8	4	3	$\boxed{0}$	
Dummy	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	

(d)

Step 5 : Test for optimality :

Neither the third row nor the third column has any assignment.

Step 6 : By drawing the four lines along the first, fourth and fifth rows and the first column, all the zeros can be covered. Refer to the matrix (d) above.

It may be done by the following method also.

(a) The third row has no assignment. It is ticked.

(b) It has the crossed zero in the first column which is ticked.

(c) The first column has assigned zero in the second row which is ticked.

(d) No repetition of (b) and (c) is possible as the second row has no crossed zero.

(e) The first, fourth and fifth rows are unticked. The first column is ticked. Hence, lines are drawn along them.

Step 7 : In the matrix (d), 4, 3, 7, 5, 1, 4, 7 and 9 are the uncovered elements. The least among them is 1. Hence, 1 is subtracted from them and is added to 8, 6 and 0 which are at the intersections of the lines. The other elements along the lines 9, 2, 0, 6, 0, 0, 8, 4, 3, 0, 0, 0, 0 and 0 remain unaltered. They are given in the revised matrix (e) below.

Machine

Job	I	II	III	IV	V
A	9	9	2	0	6
B	0	3	2	6	4
C	0	0	3	6	8
D	7	8	4	3	0
Dummy	1	0	0	0	0

(e)

Machine

Job	I	II	III	IV	V
A	9	9	2	$\boxed{0}$	6
B	$\boxed{0}$	3	2	6	4
C	$\cancel{0}$	$\boxed{0}$	3	6	8
D	7	8	4	3	$\boxed{0}$
Dummy	1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

(f)

Step 8 : The assignments are made as in step 4 and shown in the matrix (f) above.

There is only one zero each in the first two rows. Squares are put around those zeros. The other zeros in the fourth and first columns are crossed. Only one zero is left in the third row. A square is put around it. The other zero in the second column is crossed. A square is put around the zero in the fourth row. The other zero in the fifth column is crossed. Only one zero is left blank at this stage. A square is put around it.

The optimal assignment schedule :

A → IV; B → I ; C → II; D → V

(Machine III is kept idle.)

The minimum assignment (total time) =

$$12 + 17 + 16 + 17 = 62 \text{ hours.}$$

Example 5 : (Maximization Assignment Problem)

Find the maximum profit from the following assignment problem. Profits are given in rupees.

Machine

Job	A	B	C	D	E
1	17	23	25	13	25
2	25	9	13	6	21
3	26	12	18	15	22
4	7	23	26	21	21
5	14	18	25	20	24

Solution : The maximization problem can be solved by any one of the following two methods.

Method I : All the elements are subtracted from the greatest element 26 to get the opportunity losses. The problem is then solved as in minimization problems.

Machine

Job	A	B	C	D	E
1	9	3	1	13	1
2	1	17	13	20	5
3	0	14	8	11	4
4	19	3	0	5	5
5	12	8	1	6	2

Opportunity Losses

Machine

Job	A	B	C	D	E
1	8	2	0	12	0
2	0	16	12	19	4
3	0	14	8	11	4
4	19	3	0	5	5
5	11	7	0	5	1

(a)

Step 1 : The number of rows = The number of columns. It is a balanced problem. It is solved as follows :

Step 2 : Reduce the elements of each row :

1, 1, 0, 0 and 1 are the least elements in the respective rows. 1 is subtracted from the elements of the first row, 1, from those of the second row. . . . The reduced matrix (a) is given above.

Step 3 : Reduce the elements of each column:

0, 2, 0, 5 and 0 are the least elements in the respective columns. 0 is subtracted from the elements of the first column, 2, from those of the second column . . . The reduced matrix (b) is given below.

Machine

Job	A	B	C	D	E
1	8	0	0	7	0
2	0	14	12	14	4
3	0	12	8	6	4
4	19	1	0	0	5
5	11	5	0	0	1

(b)

Machine

Job	A	B	C	D	E
1	8	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$
2	$\boxed{0}$	14	12	14	4
3	$\cancel{0}$	12	8	6	4
4	19	1	$\boxed{0}$	$\cancel{0}$	5
5	11	5	$\cancel{0}$	$\boxed{0}$	1

(c)

Step 4 : Make the assignments :

A square is put around the zero in the second row. The other zero in the first column is crossed. A square is put across the zero in the second column. The other zeros in the first row are crossed. There are two zeros each in the fourth and fifth rows. Squares are put across two zeros arbitrarily and the two other zeros are crossed. They are shown in the matrix (c) given above.

Step 5 : Test for optimality :

Neither the third row nor the fifth column has any assignment.

Step 6 : (a) The third row has no assignment. It is ticked.

(b) In the third row, crossed zero is in the first column. The first column is ticked.

(c) In the first column, assigned zero is in the second row. The second row is ticked.

(d) No repetition of (b) and (c) is possible as the second row has no crossed zero.

(e) The first, fourth and fifth rows are unticked. The first column is ticked. Hence, lines are drawn along them.

They are shown in the matrix (d) given below.

Job	A ✓	B	C	D	E	
1	8	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$	
2	$\boxed{0}$	14	12	14	4	✓
3	$\cancel{0}$	12	8	6	4	✓
4	19	1	$\boxed{0}$	$\cancel{0}$	5	
5	11	5	$\cancel{0}$	$\boxed{0}$	1	

Machine

(d)

Job	A	B	C	D	E
1	12	0	0	7	0
2	0	10	8	10	0
3	0	8	4	2	0
4	23	1	0	0	5
5	15	5	0	0	1

Machine

(e)

Step 7 : In the matrix (d), 14, 12, 14, 4, 12, 8, 6 and 4 are the uncovered elements. 4 is the least among them. It is subtracted from all those elements and is added to 8, 19 and 11 which are at the intersections of the lines. The other elements 0, 0, 7, 0, 0, 0, 1, 0, 0, 5, 5, 0, 0 and 1 are left unchanged. They are given in the matrix (e) above.

Step 8 : The assignments are made as in step 4 and shown in the matrices given below.

In no row, there is one zero. In the second column, there is one zero. A square is put around it. The other two zeros in the first row are crossed.

As there are two zeros in the remaining rows and columns, the subsequent assignments are made arbitrarily. Hence, the following alternative optimal assignments :

1. Job — Machine

Job	A	B	C	D	E
1	12	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$
2	$\boxed{0}$	10	8	10	$\cancel{0}$
3	$\cancel{0}$	8	4	2	$\boxed{0}$
4	23	1	$\boxed{0}$	$\cancel{0}$	5
5	15	5	$\cancel{0}$	$\boxed{0}$	1

$1 \to B$; $2 \to A$; $3 \to E$; $4 \to C$; $5 \to D$

Max. $Z = 23 + 25 + 22 + 26 + 20 =$ Rs. 116.

2. Job   Machine

Job	A	B	C	D	E
1	12	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$
2	$\boxed{0}$	10	8	10	$\cancel{0}$
3	$\cancel{0}$	8	4	2	$\boxed{0}$
4	23	1	$\cancel{0}$	$\boxed{0}$	5
5	15	5	$\boxed{0}$	$\cancel{0}$	1

1 → B; 2 → A; 3 → E; 4 → D; 5 → C.

Max. Z = 23 + 25 + 22 +21 + 25 = Rs. 116

3. Job   Machine

Job	A	B	C	D	E
1	12	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$
2	$\cancel{0}$	10	8	10	$\boxed{0}$
3	$\boxed{0}$	8	4	2	$\cancel{0}$
4	23	1	$\boxed{0}$	$\cancel{0}$	5
5	15	5	$\cancel{0}$	$\boxed{0}$	1

1 → B; 2 → E; 3 → A; 4 → C; 5 → D.

Max. Z = 23 + 21 + 26 + 26 + 20 = Rs. 116.

4. Job   Machine

Job	A	B	C	D	E
1	12	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$
2	$\cancel{0}$	10	8	10	$\boxed{0}$
3	$\boxed{0}$	8	4	2	$\cancel{0}$
4	23	1	$\cancel{0}$	$\boxed{0}$	5
5	15	5	$\boxed{0}$	$\cancel{0}$	1

1 → B; 2 → E; 3 → A; 4 → D; 5 → C.

Max. Z = 23 + 21 + 26 + 21 + 25 = Rs. 116.

Method II : All the elements are multiplied by –1. The problem is then solved as in a minimization problem.

The given matrix after its elements are multiplied by –1:

Machine

Job	A	B	C	D	E
1	–17	–23	–25	–13	–25
2	–25	– 9	–13	– 6	–21
3	–26	–12	–18	–15	–22
4	– 7	–23	–26	–21	–21
5	–14	–18	–25	–20	–24

Step 1 : The number of rows = The number of columns. It is a balanced problem. It is solved as follows :

Step 2 : Reduce the elements of each row :

–25, –25, –26, –26 and –25 are the least elements in the respective rows. –25 is subtracted from the elements of the first row, –25, from those of the second row . . . . The reduced matrix (a) is found below :

Machine

Job	A	B	C	D	E
1	8	2	0	12	0
2	0	16	12	19	4
3	0	14	8	11	4
4	19	3	0	5	5
5	11	7	0	5	1

(a)

This is the same matrix which was obtained in step 2 by Method 1. Hence, the subsequent steps and the solutions are the same as in Method 1.

Example 6 : (Restricted Assignment Problem)

Solve the following assignment problem (operator 1 cannot be assigned to machine C and operator 3 cannot be assigned to machine D) :

		Machine			
		A	B	C	D
Operator	1	50	50	-	20
	2	70	40	20	30
	3	90	30	50	-
	4	70	20	60	70

(B.Com. Madras, O 2K)

Solution : In minimization problems, the costs at the restricted cells are infinity (∞). When any finite cost is subtracted from them, they remain infinity. Hence, there is no possibility of getting zeros in those cells.

The procedure is similar to that of unrestricted assignment problems.

Step 1 : The number of rows = The number of columns. It is a balanced problem. It is solved as follows :

Step 2 : Reduce the elements of each row :

20, 20, 30 and 20 are the least elements in the respective rows. 20 is subtracted from the elements of the first row, 20, from those of the second row . . . The reduced matrix (a) is given below.

Operator	Machine A	B	C	D
1	30	30	-	0
2	50	20	0	10
3	60	0	20	-
4	50	0	40	50

(a)

Operator	Machine A	B	C	D
1	0	30	-	0
2	20	20	0	10
3	30	0	20	-
4	20	0	40	50

(b)

Step 3 : Reduce the elements of each column :

30, 0, 0 and 0 are the least elements in the respective columns. 30 is subtracted from the elements of the first column. As 0 is subtracted from the elements of the other columns, there is no change in them. The reduced matrix (b) is given above.

Step 4 : Make the assignments :

Squares are put around the zeros in the second and third rows. The other zero in the second column is crossed. A square is put around the zero in the first column. The other zero in the first row is crossed. They are shown in the matrix (c) given below.

Operator Machine

$$\begin{array}{c|cccc|}
 & A & B & C & D \\
\hline
1 & \boxed{0} & 30 & - & \cancel{0} \\
2 & 20 & 20 & \boxed{0} & 10 \\
3 & 30 & \boxed{0} & 20 & - \\
4 & 20 & \cancel{0} & 40 & 50 \\
\hline
\end{array}$$

(c)

Operator Machine

$$\begin{array}{c|cccc|c}
 & A & B & C & D & \\
 & & \checkmark & & & \\
\hline
1 & \boxed{0} & 30 & - & \cancel{0} & \\
2 & 20 & 20 & \boxed{0} & 10 & \\
3 & 30 & \boxed{0} & 20 & - & \checkmark \\
4 & 20 & \cancel{0} & 40 & 50 & \checkmark \\
\hline
\end{array}$$

(d)

Step 5 : Test for optimality :

Neither the fourth row nor the fourth column has any assignment.

Step 6 : (a) The fourth row has no assignment. It is ticked.

(b) In the fourth row, crossed zero is in the second column. The second column is ticked.

(c) In the second column, assigned zero is in the third row. The third row is ticked.

(d) In the third row, there is no crossed zero.

(e) The first and the second rows are unticked. The second column is ticked. Lines are drawn along them.

They are shown in the matrix (d) given above.

Step 7 : In the matrix (d), 30, 20, -, 20, 40 and 50 are the uncovered elements. 20 is the least among them. 20 is subtracted from them and is added to 30 and 20 which are at the intersections of the lines. The other elements 0, -, 0, 20, 0, 10, 0 and 0 remain unaltered. They are shown in the matrix (e) given in the next page.